Speed of thermal expansion of a long, thin insulating bar and the physical momentum of acoustic phonons

Speed of thermal expansion of a long, thin insulating bar and the physical momentum of acoustic phonons

Y C Lee

Physics Department, University at Buffalo, SUNY, Amherst, NY 14260, USA

Abstract
Thermal expansion is an everyday phenomenon. One would naturally be curious to see how fast the expansion proceeds. While the theory of thermal expansion in statistical thermal equilibrium is well known, the *time-dependent process* during thermal expansion is a more complex statistical dynamical problem. Contrary to intuitive expectations, it will be seen that the dynamical expansion process is generally different from the process of merely establishing temperature equilibration (thermal-kinetic equilibrium) because two vastly disparate timescales are at work. It will be shown that the finite speed of thermal expansion hinges upon a recently derived result that an acoustic phonon of wavevector $\vec{q} \neq 0$ does carry a finite physical momentum; it arises from anharmonicity, provided translational symmetry is broken. While the eventual mathematical formulation seems pedestrian, it is arrived at after several layers of physical thinking. Our final result shows that the *time* required for thermal expansion of a thin bar of length L by ΔL due to a given temperature increase ΔT is given by $\Delta t_L \propto (L/\Delta L)$ (L/c_s), where c_s is the speed of sound. Its physical origin as well as its classical and quantum limits are fully discussed.

1. Introduction

As shown in the paper 'Physical momentum versus crystal momentum of acoustic phonons in a crystal lattice' [1], in addition to the more familiar 'crystal momentum' an acoustic phonon in a crystal lattice also possesses a physical momentum. As we shall see, the present paper provides an important application of the concept of the physical momentum of an acoustic phonon in a crystal lattice.

Thermal expansion is an everyday phenomenon. How a bar becomes lengthened when it is heated is not only a matter of scientific curiosity but should be of great practical interest as well. Our problem at hand is, therefore, the speed of thermal expansion in progress rather than thermal expansion as an accomplished act. Although thermal expansion is a familiar topic in standard solid-state texts [2], there seems to be hardly any work on how fast the expansion proceeds. This is because the former is a problem of statistical thermal equilibrium but the latter belongs to the more complex realm of statistical non-equilibrium thermal dynamics. On the other hand, it will be seen that familiar concepts in the former such as the heat capacity, the Grüneisen constant γ, the thermal conductivity, the speed of sound c_s and so on will make their way into the

investigation of the latter as well. In fact, it will be shown that the time Δt_L required for the thermal expansion of a long thin bar of length L by ΔL due to a given temperature rise ΔT is given by $\Delta t_L \sim (L/\Delta L)(L/c_s)$.

By carefully re-interpreting the statistical mechanics of the thermal expansion coefficient [2] and by calculating explicitly the time of heat diffusion transversely into a thin crystal bar, two distinct timescales in the thermal expansion process will be identified in section 2. Based on these two timescales a first attempt at formulating an equation to calculate the thermal expansion as a function of time is made in section 3. However, a serious flaw is found in that the equation for it fails to predict a steady state for the thermal expansion. Instead, an oscillatory expansion is found. It is at this crucial stage that the concept of physical momentum of an acoustic phonon [1] enters. It provides a steadying mechanism. A brief review of the physical momentum of acoustic phonons is given in section 4. An approach with a finite average speed to a final steady value of thermal expansion is subsequently obtained in section 5. Physical discussions of this result as well as its quantum and classical limits are given in section 6. A summary and conclusion is given in the final section 7.

2. Two distinct timescales

Generally the problem of approach to equilibrium of a system possessing many kinds of interacting degrees of freedom may involve several widely disparate timescales. Our problem at hand is the time rate of thermal expansion. Naively one might assume that as soon as a specimen has been uniformly heated to a higher temperature by ΔT, the thermal expansion to a larger volume by ΔV would be simultaneously accomplished. That this is false can be seen as follows. Imagine a long, moderately narrow and very thin slab of thickness b and length L initially placed in contact with a heat reservoir such as a huge pot of water at temperature T. When we scoop it up and immediately submerge it into another huge pot of water at a higher temperature $T + \Delta T$, the time Δt_T for the thin slab to warm up uniformly to that higher temperature can be obtained by solving the heat diffusion equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \tag{1}$$

for the distribution of the temperature $T(x,t)$ as a function of time t and position x across the transverse dimension or the thickness of the slab. Here $\kappa = \frac{K}{C}$, where K is the thermal conductivity and C the heat capacity [3]. The exact solution is

$$T(x,t) = (T + \Delta T) \left[1 - \frac{4}{\pi} \sum_{l=1,3,5,\dots} \frac{1}{l} \sin \frac{l\pi x}{b} e^{-\kappa \frac{\pi^2 l^2}{b^2} t} \right]. \tag{2}$$

The $l = 1$ term in the sum dominates the long time behavior. Accordingly the slab approaches a thermally steady state in which the entire slab attains the uniform temperature $(T + \Delta T)$ independent of x when $t \gg \Delta t_T$, where

$$\Delta t_T \approx \frac{1}{\kappa \pi^2} b^2. \tag{3}$$

In other words, Δt_T is the timescale for temperature equilibration or 'thermal-kinetic equilibrium'. Microscopically, such a temperature equilibration is achieved by the *diffusion* of kinetic energy via interparticle collisions, recalling especially the kinetic definition of temperature that pertains to the translational degrees of freedom [4]. That this Δt_T of equation (3) turns out to be independent of ΔT is not surprising. Both the rate of heat conduction $\frac{\Delta Q}{\Delta t} = KA\Delta T/b$ and the amount of energy $\Delta Q = CAb\Delta T$ to be transported from the surrounding heat reservoir to the slab are proportional to ΔT, thereby corroborating $\Delta t \sim \frac{Cb^2}{K} = \frac{b^2}{\kappa}$. (For a quartz slab at $T = 0\,°C$, $C = 0.48$ cal cm^{-3} $°$C^{-1}, $K = 0.03$ cal cm^{-1} $°$C^{-1} s^{-1}, $\kappa = 16$ s cm^{-2}; for aluminum, $\kappa = 1.18$ s cm^{-2}.) However, no statement can be made by the diffusion equation (1) about the mechanical increase in length.

On the other hand, it will be shown later that the time for thermal expansion of a long thin slab is proportional to L, being given by $\Delta t_L \approx (L/c_s)/(9\alpha \Delta T)$ of equation (19) where c_s is the sound speed and α the thermal expansion coefficient. We could imagine the length L to be very large compared to the thickness b so that $\Delta t_L / \Delta t_T \approx (\frac{L^2}{b^2})(\frac{K}{C})(1/c_s \Delta L) \gg 1$, where $\Delta L = L\alpha \Delta T$ is the expected thermal expansion. Thus

the timescale for temperature equilibration or 'thermal-kinetic equilibrium' would generally be quite different from that for thermal expansion or 'thermal-mechanical equilibrium'. Only when both equilibria have been attained can we say that true or total thermal equilibrium is reached. Let us now address the problem of finding Δt_L.

3. A first attempt at finding Δt_L

A numerical approach might treat a linear chain of atoms connected by anharmonic springs. Based on the premise of $\Delta t_T / \Delta t_L \ll 1$ we assume that for $t \gtrsim \Delta t_T$ each atom has already received the same boost in kinetic energy proportional to ΔT from a heat reservoir but the directions of the corresponding increase in velocity are randomized [4]. We can then solve this ensemble of coupled nonlinear Newton equations for the atoms and follow the motion of each numerically. We hope to find, upon averaging over the random directions, the eventual expansion of the chain expressed by the averaged positions of the last atoms at the free ends as time passes. Our effort turned out to be a failure, for although we could trace these atomic motions numerically for a short enough chain with one end fixed, the expected orderly fractional increase $\Delta L/L$ in chain length L was only of order 10^{-5} K^{-1}, much smaller than the fractional atomic vibrations $\Delta a/a \sim 10^{-2}$, a being the lattice spacing. Trying to identify spatially this tiny $\Delta L/L$ by ensemble averaging over a sea of much larger but almost completely random atomic excursions is like finding a needle in a haystack. This difficulty is further compounded by the disparate temporal scales: Δa oscillates at a rate approximating to the Debye frequency ω_D while ΔL proceeds at the much slower pace of the thermal expansion of a macroscopic object.

A Boltzmann differential–integral equation approach [5] involving the usual semi-classical distribution function $f(\mathbf{x}, \mathbf{p}, t)$ of the atoms would not be any easier either, for it not only has to cope with the anharmonic interactions between the atoms but would also be plagued by the same disparities in the scales of space and time.

To look for a clue elsewhere we first review the statistical mechanics of the thermal expansion coefficient [2]. Let us start with the expression for the ambient pressure P of the system at temperature T:

$$P = -\left[\frac{\partial}{\partial V} \left(U_{\text{stat.latt.}}^{\text{eq}} + \frac{1}{2} \sum_{\vec{\mathbf{q}}} \hbar \omega_{\vec{\mathbf{q}}} \right) \right]_T + P_{\text{ph}} \tag{4}$$

where

$$P_{\text{ph}} = -\left(\frac{\partial U_{\text{phonon}}}{\partial V} \right)_S = -\sum_{\vec{\mathbf{q}}} \left(\frac{\partial \hbar \omega_{\vec{q}}}{\partial V} \right)_S \langle n_{\vec{\mathbf{q}}}(T) \rangle \tag{5}$$

is the contribution to the pressure by the phonons, $\langle n_{\vec{\mathbf{q}}}(T) \rangle$ being the thermal averaged number of $\vec{q}$-phonons. On the other hand, the first term in equation (4) is the contribution from the energy of the strained lattice in static equilibrium, including that of the zero-point fluctuations. Neglecting the small ambient (e.g. atmospheric) pressure P we see that equation (4)

expresses the mechanical balance of the (positive) outward pressure of P_{ph} by the force of the stretched atomic springs that tend to restore the atoms to their original configuration. Although this equation is supposed to describe only the time independent state of total thermal equilibrium at temperature T after the full thermal expansion has been attained, we may now turn the aforementioned troublesome disparity in the two timescales into an advantage by *re-interpreting* equation (4) as follows. Based on the kinetic definition of temperature which relates it to the random kinetic energy of the atoms (i.e. of the phonons) [4] the increase of P_{ph} by ΔP_{ph} should be attained concurrently with the increase of temperature by ΔT. Once these increases have been accomplished within the short timescale Δt_T of equation (3), the thermal expansion of the long thin bar, still submerged in the huge pot of hotter water (or thermal reservoir) at $T + \Delta T$, would proceed at the much slower timescale Δt_L *isothermally*. At every subsequent stage of the expansion process the lattice may be viewed as consisting of a quasi-static, strained lattice plus the very fast lattice vibrations (phonons) about this quiescent lattice, corresponding to the first and the second term of equation (4), respectively. At each such quasi-static stage, the thermodynamic variables T, P and V can thus be assumed to have well-defined values. However, unlike the total thermal equilibrium in which the two terms balance each other, the outward phonon force of the second term would now be too large to be balanced by the restoring force of the strained atomic springs until the thermal expansion has finally been completed. This is of course why the system keeps expanding until thermo-mechanical equilibrium is reached.

In this re-interpretation the problem has become that of a quiescent elastic lattice of cross section A, being pulled outward by ΔP_{ph} at every quasi-static stage of the slow isothermal expansion process. All the random anharmonic lattice vibrations about this quiescent lattice have now been thermally averaged to appear only in this $\Delta P_{ph} \propto \gamma$, based on equation (5) in the usual approximation of $\partial \omega_{\vec{q}} / \partial V = -\gamma \omega_{\vec{q}} / V$, γ being the Grüneisen coefficient [2]. Other anharmonic effects such as phonon–phonon interaction are neglected by treating the phonons as a gas of free bosons under this Grüneisen approximation, rendering the harmonic lattice vibration modes still independent of each other [2].

Without loss of generality consider a long, thin crystalline bar of cross section A with one end fixed, for a bar with both ends free is equivalent to a bar with the center of mass fixed during thermal expansion. Suppose that under the above-mentioned pull of ΔP_{ph} every spring between two neighboring atoms separated by lattice spacing a is stretched by the same averaged amount $\langle x(t) \rangle$. Thus the nth layer of atoms along this bar, counting from the fixed end, would be displaced by $n \langle x(t) \rangle$, and the layer at the free end of this bar of N layers by $\Delta L(t) = N \langle x(t) \rangle$. With ΔP_{ph} being uniform within the bar, there is no net phonon force acting on any internal layer of atoms except at the free end. Conservation of energy then leads to $A \Delta P_{ph} N \langle x(t) \rangle = (NA/a^2)(\frac{k}{2} \langle x(t) \rangle^2) + \frac{1}{2}(mA/a^2) \sum_{n=1}^{N} (nv)^2$. Differentiating with respect to t then

yields the equation of motion for a whole layer of atoms,

$$\frac{d^2 \langle x(t) \rangle}{dt^2} + \omega_e^2 \langle x(t) \rangle = a^2 \frac{\Delta P_{ph}}{m_{eff}} \qquad (6)$$

where

$$\omega_e^2 = \frac{k}{m_{eff}} \equiv \frac{3k}{mN^2} \qquad (7)$$

k being the atomic spring constant and m the atomic mass. *Note that classical physics suffices because the quantum mechanical averaged motion of a whole layer of atoms is macroscopic.* This accounts for the simplicity of this approach.

The frequency ω_e is the natural frequency of oscillation for each layer of atoms when the long thin bar is stretched momentarily by any force and then let go. This frequency is purely mechanical in nature due completely to the stiffness of the atomic springs. It is completely unrelated to, and hence would not interfere with, any thermal or statistical processes. Whether its magnitude is large or small compared to $1/\Delta t_T$ of equation (3) would neither affect nor be affected by, for example, the temperature equilibration process across the transverse dimension of the thin bar described by equation (2). In other words, by themselves the longitudinal oscillations of the long thin bar set up by an external perturbation of whatever cause would not generate any entropy of their own. Furthermore, any slow, quasi-static thermal expansion lengthwise which presumably would generate entropy is simply superimposed on the mechanical oscillation. This is why the timescale of ω_e^{-1} is not of significance and was not included in the discussion in section 2.

The solution to equation (6) is $\Delta L(t) = (Na^2 \Delta P_{ph}/k)(1 - \cos \omega_e t)$. This means, however, that the thermal expansion would not settle down due to some damping process to a steady value $\Delta L = a^2 \Delta P_{ph}/(k/N)$ but would rather oscillate with an effective frequency ω_e about such a value. This is thermo-dynamically unacceptable. Clearly some key physical mechanism that could provide a damper to the oscillatory behavior is missing. But there is no entropy associated with a quasi-static lattice nor with a mechanically vibrating one. And we have neglected phonon–phonon scattering. Where else can we find a source of dissipation that leads to the damper?

4. Physical momentum of an acoustic phonon

To this end we re-examine more critically the concept of the phonon pressure of equation (5), evaluated now in the Grüneisen approximation:

$$P_{ph} = \frac{\gamma \sum_{\vec{q}} \hbar \omega_{\vec{q}} \langle n_{\vec{q}}(T) \rangle}{V} = \frac{\gamma U_{phonon}(T)}{V} \qquad (8)$$

where $U_{phonon}(T)/V$ is the energy density of the phonon gas at temperature T. As in the case of the more familiar photon pressure we may adopt a purely kinetic-theory point of view to think of the phonon pressure as due to the bombardments by the bullet-like phonons. This implies inevitably that an acoustic phonon of frequency $\omega_q = c_s q$ and wavevector $\vec{q}$ should carry a physical momentum proportional to $\vec{q}$. Otherwise, how else can it give rise to any force or pressure?

Indeed it has been most recently shown that an acoustic phonon could actually carry a physical momentum, provided translational invariance is broken, as for our crystalline bar of finite length that is in contact with an external heat reservoir [1]. The broken translational symmetry enables an acoustic phonon of $\vec{q} \neq 0$ to couple (via anharmonic interaction) to another acoustic phonon mate of $\vec{q} = 0$, thereby forming a composite phonon of the same $\vec{q}$. As is well known, the $q = 0$ acoustic mode corresponds to a uniform translation of the lattice. Thus it is this mate of $q = 0$ that endows a physical momentum to the composite phonon. In fact, it was proved that in processes involving broken translational invariance an acoustic $\vec{q}$-phonon in n-dimensions behaves like a corpuscular particle with a physical momentum [1]

$$\vec{p}_{\vec{q}} = n\hbar\gamma\vec{q}. \tag{9}$$

This momentum would vanish in the absence of anharmonic coupling when $\gamma = 0$. It should be distinguished from the more familiar 'crystal momentum' $\vec{p}_{\vec{q}}^{\text{crystal}} = \hbar\vec{q}$ in any dimension (see, for example [2, 6, 7]), which is operative in processes respecting translational crystalline symmetry such as electron–phonon scattering within the crystalline medium [1].

5. The time for thermal expansion Δt_L

We now proceed along the kinetic-theoretic line of thinking to re-calculate the pressure of the bullet-like phonons impinging on the free end of a thin and long crystal bar (of cross-sectional area A) that is fixed at the other end. But this time we take into account the velocity $c_s - v_{\text{end}}$ of the phonons relative to the free end that is expanding outward with velocity v_{end}. Thus the phonon frequency $\omega_{\vec{q}}$ is Doppler shifted to $\omega'_{\vec{q}} = \omega_{\vec{q}}(c_s - v_{\text{end}})/c_s$. Correspondingly the momentum of each $\vec{q}$-phonon impinging on the expanding free end is changed from $\vec{p}_{\vec{q}}$ of equation (9) to the relative momentum $\vec{p}'_{\vec{q}}$ along the longitudinal direction:

$$\vec{p}'_{\vec{q}} = \frac{1\hbar\gamma\omega'_{\vec{q}}}{c_s} = \left(\frac{1\hbar\gamma\omega_{\vec{q}}}{c_s}\right)\left(\frac{c_s - v_{\text{end}}}{c_s}\right)$$

$$= \vec{p}_{\vec{q}}\left(1 - \frac{v_{\text{end}}}{c_s}\right). \tag{10}$$

The accompanying phonon pressure is changed from P_{ph} of equation (8) to

$$P'_{\text{ph}} = \sum_{\vec{q}}\langle n_{\vec{q}}/2V\rangle(c_s - v_{\text{end}})2p'_{\vec{q}} = P_{\text{ph}}(T) + \delta P_{\text{ph}}(v_{\text{end}}) \tag{11}$$

which represents the change of momentum per unit time due to the bombardments of the phonons of density $\langle n_{\vec{q}}/2V\rangle$ on a unit normal area of the outward expanding (with speed v_{end}) free end. The phonon pressure when $v_{\text{end}} = 0$ is given by

$$P_{\text{ph}}(T) = \sum_{\vec{q}}\langle n_{\vec{q}}/2V\rangle c_s 2p_{\vec{q}} = \sum_{\vec{q}}\langle n_{\vec{q}}(T)/V\rangle\gamma\hbar\omega_{\vec{q}} \tag{12}$$

and

$$\delta P_{\text{ph}}(v_{\text{end}}) \approx -\frac{2v_{\text{end}}}{c_s}P_{\text{ph}}(T) \tag{13}$$

is the correction to the phonon pressure due to the Doppler shift. While this correction varies directly with the prevailing temperature-dependent phonon pressure $P_{\text{ph}}(T)$ itself, most significantly it is proportional linearly but directed oppositely to the expanding velocity v_{end}. Hence it provides a damping force that is the missing link we have been looking for. It owes its existence entirely to the bombardment by the nonvanishing phonon momentum of equation (9). In turn, the existence of the phonon momentum itself is rendered possible because the requirement of broken translational symmetry is met by the externally imposed ΔT from the heat reservoir. Thus the obstacle to acquiring a finite momentum by a $\vec{q} \neq 0$ phonon via the Grüneisen coupling to its acoustic mate of $q = 0$ is removed [1]. (In contrast, water pressure on a receding vertical wall due to gravitation would not have such a δP of Doppler origin as in equation (13), for that pressure arises not from discrete corpuscular bombardments but rather from the gravitational forces transmitted sideway through a nearly incompressible continuum.)

Energy is dissipated in every collision of the phonon with the outward expanding free end because, in the laboratory frame, the reflected momentum is smaller in magnitude than the incident momentum. The greater the phonon pressure that is proportional to the product of the temperature-dependent phonon density and the difference between the forward and the reflected phonon momenta of each phonon upon collision with the expanding free end, the greater will be the energy dissipation.

As the temperature rises by ΔT the Doppler-shift correction would itself change due to ΔT according to equation (13) by

$$\Delta[\delta P_{\text{ph}}(v_{\text{end}})] \approx -\frac{2v_{\text{end}}}{c_s}\Delta P_{\text{ph}}(T). \tag{14}$$

With this correction taken into account our previous equation of motion (6) is now modified, with $v_{\text{end}} = N\,\mathrm{d}\langle x(t)\rangle/\mathrm{d}t$, to become

$$\frac{\mathrm{d}^2\langle x(t)\rangle}{\mathrm{d}t^2} + \omega_e^2\langle x(t)\rangle + \eta\frac{\mathrm{d}\langle x(t)\rangle}{\mathrm{d}t} = a^2\frac{\Delta P_{\text{ph}}}{m_{\text{eff}}} \tag{15}$$

where

$$\eta = \frac{6a^2}{Nmc_s}\Delta P_{\text{ph}}(T) \tag{16}$$

is the damping coefficient that we are looking for. Here $a^2\Delta P_{\text{ph}}(T)$ is the increase of phonon force due to ΔT, pushing outward at the free end of every linear *chain* of atoms whose total mass is Nm within the thin bar.

Following equation (12) we find

$$\Delta P_{\text{ph}}(T) = -\sum_{\vec{q}}(\partial\hbar\omega_{\vec{q}}/\partial V)_s\Delta\langle n_{\vec{q}}(T)\rangle$$

$$= \gamma\sum_{\vec{q}}(\hbar\omega_{\vec{q}}/V)\Delta T\frac{\partial}{\partial T}\langle n_{\vec{q}}(T)\rangle$$

$$= [\gamma(\partial U/\partial T)_V/V]\Delta T = \gamma c_v\Delta T \tag{17}$$

where $c_v = (\partial U/\partial T)_V/V$ is the heat capacity. This $\Delta P_{\text{ph}}(T)$ is directly proportional to the change of the phonon density and hence to the rise in temperature ΔT as it should be from

the kinetic definition of temperature [4]. As a result we obtain from equation (16)

$$\eta = 6\gamma c_v \Delta T / (L c_s \rho) \qquad (18)$$

in which $\rho = m/a^3$ is the mass density. Note that η is independent of ω_e, which is consistent with the discussion near the end of section 3. Comparing with the thermal expansion coefficient [2] $\alpha = \gamma c_v/3B = \gamma c_v/(3\rho c_s^2)$ we finally obtain

$$\Delta t_L \equiv \frac{2}{\eta} = \frac{1}{9}\left(\frac{L}{c_s}\right)\left(\frac{L}{\Delta L}\right) = \frac{1}{9}\left(\frac{L}{c_s}\right)\frac{1}{\alpha \Delta T}. \qquad (19)$$

Counter to naive expectation Δt_L is inversely proportional to ΔT. Mathematically this is due to the fact that the Doppler-shift correction to $\Delta P_{ph}(T)$ (caused by temperature rise ΔT) as given by equation (14) is proportional to $\Delta P_{ph}(T)$ and hence to ΔT itself. Physically this is consistent with our previous discussion of the relation between energy dissipation and the phonon pressure that is essentially the product of the temperature-dependent phonon density and the difference between the forward and the reflected momenta of the phonons as each of them bombards the outward-expanding free end of the bar. The expanding motion is now

$$\Delta L(t) \approx \Delta L \left[1 - e^{-\frac{\eta}{2}t}\left(\cos\omega_e t + \frac{\eta}{2\omega_e}\sin\omega_e t \right) \right] \qquad (20)$$

where we have neglected higher order terms in $\eta/\omega_e \ll 1$. Here $\Delta L = \alpha L \Delta T$ is the final value of thermal expansion. It is seen that the length of the long thin bar oscillates with a frequency ω_e as it slowly expands on its way to the final value at the rate of $1/\Delta t_L$ given by equation (19). The back and forth oscillations are due, of course, to the inertia of the motion as the bar is pulled outward by the force that is proportional to the right side of equation (15).

6. Physical discussion

The fact that η of equation (18), like α, is proportional to $c_v(T)$ means that it is independent of T in the classical limit when $T \gg T_{\text{Debye}}$, and proportional to T^3 in the quantum limit when $T \ll T_{\text{Debye}}$. That the sound speed c_s sets the basic timescale of L/c_s in Δt_L of equation (19) is expected physically. The fact that the long bar should only expand outward thermally from the free end rather than from the fixed end as the temperature rises clearly requires the transmission of information by a messenger (the phonon) traveling at sound speed at least once from one end to the other before the atoms in the bar can tell the difference between the two ends. The other factor $L/\Delta L$ in Δt_L simply expresses the fact that it takes, on average, not merely one round but rather a multiple of $L/\Delta L$ rounds of traversal over the length L before the thermal expansion can settle down to a steady value. For a given ΔT, a smaller $\Delta L/L$ means a smaller expansion coefficient α or c_v and hence a smaller force $\Delta P_{ph}(T)$ according to equation (17). This naturally leads to a correspondingly slower speed or longer time Δt_L for the expansion.

7. Summary and conclusion

Although the eventual mathematical approach in this work seems pedestrian and elementary, it is arrived at only after several layers of physical thinking. Thus we have exploited the disparity in the two timescales, $\Delta t_L \gg \Delta t_T$, to arrive at a dynamic relation connecting the successive quasi-static stages of the *slow* process of thermal expansion of a long thin bar in which the prevailing *instantaneous* phonon pressure P_{ph} more than balances the restoring elastic force until the expansion is finally completed. To find a mechanism that would have a steadying influence on the otherwise oscillatory expansion process, we make use of a recently derived result [1, 8] for the physical momentum $\vec{p}_{\vec{q}} = n\hbar\gamma\vec{q}$ for a $\vec{q}$-phonon. As the free end of the crystalline bar expands with a finite velocity v_{end}, the momentum of the pressure-causing phonons bombarding that free end would suffer a Doppler shift, thereby resulting in a reduction δP_{ph} that is opposite and proportional to $\vec{v}_{\text{end}}$. This provides the dissipative, steadying mechanism in equation (15). This equation plays a role in the approach to thermal-mechanical equilibrium similar to that played by the heat diffusion equation (1) in the approach to thermal-kinetic equilibrium that yields Δt_T of equation (2). It is this steadying mechanism that gives rise to the thermal expansion time Δt_L of equation (19), the central result of this investigation. Its physical origin as well as its classical and quantum limits have been discussed. The rate of thermal expansion of a thin metallic bar will be published elsewhere. Other systems involving different kinds of degrees of freedom that interact in disparate timescales abound [9], as in systems involving translational motion and spin [10]. Conceivably similar ideas and techniques might perhaps be adapted to the calculation of their rates of approach to thermal equilibrium.

Some possibly related works concerning 'extended irreversible thermodynamics' [11–13] have recently been brought to our attention. For example the conceptual problems arising in the definition and measurement of temperature in situations where the local equilibrium hypothesis is no longer satisfactory were discussed in [13]. Dissipative fluxes such as heat flux q and viscous pressure tensor $\mathbf{P}^v$ (the latter is defined as a part of the pressure $\mathbf{P} = p\mathbf{U} + \mathbf{P}^v$ in which p is the equilibrium thermodynamic pressure, with $\mathbf{U}$ the identity tensor) are proposed as independent variables in this theory. Whereas the evolution equations for the classical variables u, v and $\vec{v}$ are related to conservation laws, no general criteria exist concerning the evolution equations of the dissipative fluxes. Restrictions on the form of the latter evolutions will be imposed by the second law of macroscopic thermodynamics [11, 12]. However, it is clear that the relation of these works to ours is rather remote, for we are not dealing with the evolution of any of the above-mentioned dissipative fluxes in any thermodynamic way. Instead, our work is microscopic, statistical and dynamical in nature. Thermodynamics comes in most significantly through the kinetic definition of temperature and hence its relation to pressure [4] which we argued was established in the very beginning of the expansion process during the short temporal scale of Δt_T of equation (3). It is not dependent on any hypothesis of spatial local equilibrium.

Our timescale Δt_L for thermal expansion is calculated from the dynamical equation (equation (15)) according to the response to the outward expanding force of thermal pressure that results from the Doppler-shifted momenta of the microscopic, individual, corpuscular acoustic phonons, rather than from any macroscopic constitutive equations.

Some other works concerning radiation pressure resulting from wave propagations [14] and radiation forces associated with heat propagation [15] could have interesting connection to our work. We shall, however, postpone this investigation until later.

References

[1] Lee Y C and Lee W Z 2006 *Phys. Rev.* B **74** 172303
[2] Aschcroft N W and Mermin N D 1976 *Solid State Physics* (New York: Holt, Rinehart, and Winston) chapter 25
[3] Mathews J and Walker R L 1970 *Mathematical Methods of Physics* 2nd edn (New York: Benjamin) chapter 8, p 234
[4] Feynman R P, Leighton R B and Sands N 1963 *The Feynman Lectures in Physics* vol 1 (Reading, MA: Addison-Wesley) chapter 39, section 4
[5] Peierls R E 1954 *Quantum Theory of Solids* (Oxford: Oxford University Press)
[6] Kittel C 2005 *Introduction to Solid State Physics* 8th edn (New York: Wiley) chapter 4,
[7] Ziman M 1960 *Electrons and Phonons* (London: Oxford University Press) chapter 4, p 109
[8] See also Sorbello R 1972 *Phys. Rev.* B **4757** Ceccaldi D, Ghelfenstein M and Szware H 1975 *J. Phys. C: Solid State Phys.* **8** 417
[9] Mattis D C (ed) 1993 *The Many-Body Problem* (Singapore: World Scientific)
[10] Lee Y C and Sablauer A 2005 *Phys. Rev. Lett.* **94** 130402
[11] Jou D, Casas-Vazquez J and Lebon G 1988 *Rep. Prog. Phys.* **51** 1105–79
[12] Jou D, Casas-Vazquez J and Lebon G 1999 *Rep. Prog. Phys.* **62** 1035–142
[13] Casas-Vazquez J and Jou D 2003 *Rep. Prog. Phys.* **66** 1937–2023
[14] Faber T E 1999 *Phil. Mag.* B **79** 1445–55
[15] Gaeta F S *et al* 1991 *Phys. Rev.* A **44** 5003